Progressive Wave in a Channel with Friction / and Velocity Field

Recall $\eta = \text{Re}\{E \exp(-i\omega t)\}$ Full solution

and $E = \alpha^+ \exp(ikx) + \alpha^- \exp(-ikx)$

$\longrightarrow$ incident wave $\qquad$ $\longleftarrow$ reflected wave

and $k = \frac{\omega}{c}\sqrt{1 + i\frac{R}{\omega}}$ where $c = \sqrt{gH}$

To explore effect of friction $\left(R = \frac{C_d [u]}{H}\right)$

look at solution for only incident wave

$\Rightarrow \alpha^- = 0$, and $\eta = a \cos \omega t$ at

mouth $(x=0) \Rightarrow \alpha^+ = a$ (real) [satisfies BC even if k complex]

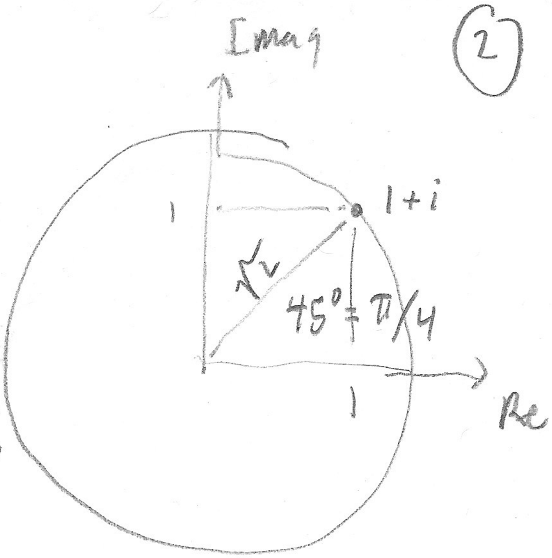
For M_2 tide $\omega = \frac{2\pi}{12.42 \text{ hours}} = 1.4 \times 10^{-4}\ s^{-1}$

and for $H = 20$ m, $[u] = 1$ m/s, $C_d = 3\times10^{-3}$

$R = \frac{C_d [u]}{H} = 1.5 \times 10^{-4}\ s^{-1} \Rightarrow$ can't neglect friction!

so take $R/\omega = 1$

Then $k = \frac{\omega}{c}\sqrt{1+i}$



Recall $1+i = 2^{1/2} \exp(i\pi/4)$

$\Rightarrow \sqrt{1+i} = 2^{1/4} \exp(i\,\pi/8)$ $22\frac{1}{2}°$

$= 2^{1/4} \cos \pi/8 + i\, 2^{1/4} \sin \pi/8$

$= 1.1 + i\, 0.46$

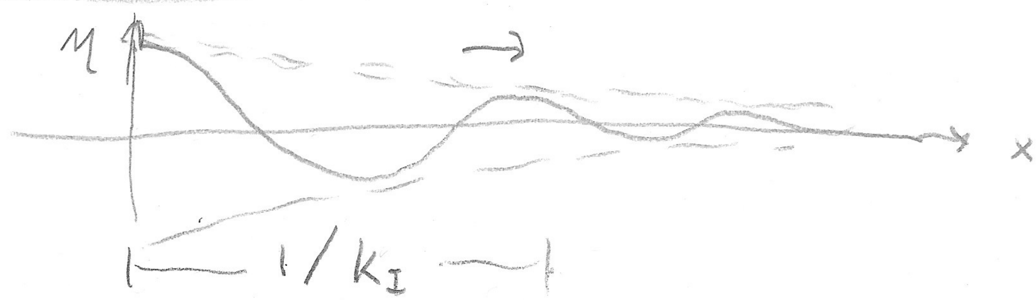
so $k = 1.1 \frac{\omega}{c} + i\, 0.46 \frac{\omega}{c}$

$= k_R + i\, k_I$

and $\eta = Re\{a \exp i(k_R x + i k_I x - \omega t)\}$

$= Re\{a\, e^{-k_I x} \exp i(k_R x - \omega t)\}$

$\Leftrightarrow$ $\boxed{\eta = a\, e^{-k_I x} \cos(k_R x - \omega t)}$



How does this compare to the frictionless solution

where $k_0 = \frac{\omega}{c_0}$, $c = \sqrt{gH}$?

$R=0 \nearrow$

(I) $\frac{k_R}{k_0} = 1.1 \Rightarrow$ 10% increase of k

$\Rightarrow$ 9% decrease of $\lambda = \frac{2\pi}{k}$

slightly shorter wavelength

(II) $\frac{c_{frictional}}{c} = \frac{\omega/k_R}{\omega/k_0} = \frac{k_0}{k_R} \Rightarrow$ 9% slower phase speed

(III) What is spatial decay scale?

$e^{-k_I x} = e^{-x/L_{decay}}$

$L_{decay} = \frac{1}{k_I} = \frac{1}{0.46\ \omega/c}$ and $c = 14$ m/s , $\omega = 1.4 \times 10^{-4}\ s^{-1}$

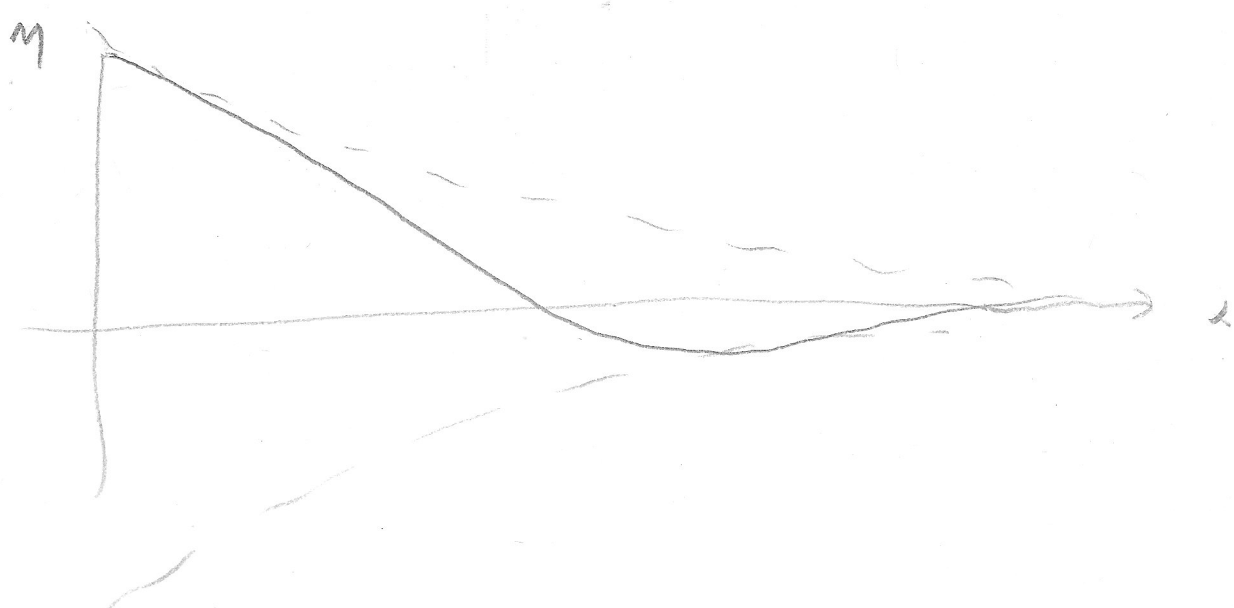
$\Rightarrow c/\omega = \frac{1}{k_0} = 1 \times 10^5$ m $= 100$ km

so $L_{decay} = 217$ km

(4)

and the wavelength $= 0.91 \dfrac{2\pi}{k_0} = 571$ km

So the solution looks like



Note: $\dfrac{1}{k}$ is a much better estimate of the spatial scale of a wave (instead of $\lambda = 2\pi/k$)

so $L \sim 100$ km

Circling back to our linearization of the SW equations, can we neglect uu_x ?

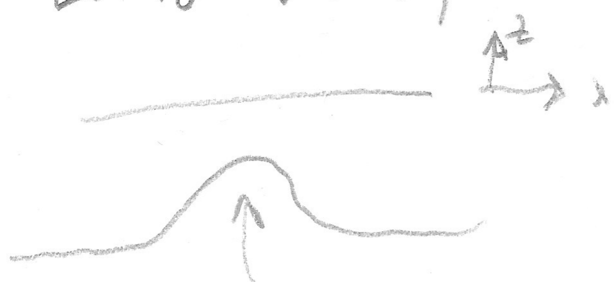
$$\frac{[uu_x]}{[u_t]} \approx \frac{[u]^2 k}{[u]\,\omega} \sim \frac{[u]}{c}$$

so for $[u] \ll c$ (recall $c = 14$ m/s for $H = 20$ m)

our linearization is good

(* Assumes $\frac{\partial}{\partial x} \sim \frac{1}{L} \approx k$ situation

is changed if L is set by bathymetry



What is the velocity field?

$\boxed{\text{mass}}$ $\quad \eta_t + H u_x = 0$

~~$u = Re\{\mathcal{U}\exp(-i\omega t)\}, \quad \eta = Re\{E\exp(-i\omega t)\}$~~

~~$\Rightarrow -i\omega E + H\mathcal{U}_x = 0$~~

~~$\Rightarrow \mathcal{U} = i\omega \int E\, dx$~~

$$\boxed{u = -\frac{1}{H}\int \eta_t \, dx}$$

$$\eta = Re\{a \exp i(kx - \omega t)\}$$

$$\Rightarrow u = Re\left\{\left(\cancel{-}\frac{1}{H}\right) a\left(\cancel{-}\cancel{i}\omega\right)\frac{1}{(\cancel{i}k)} \exp i(kx - \omega t)\right\}$$

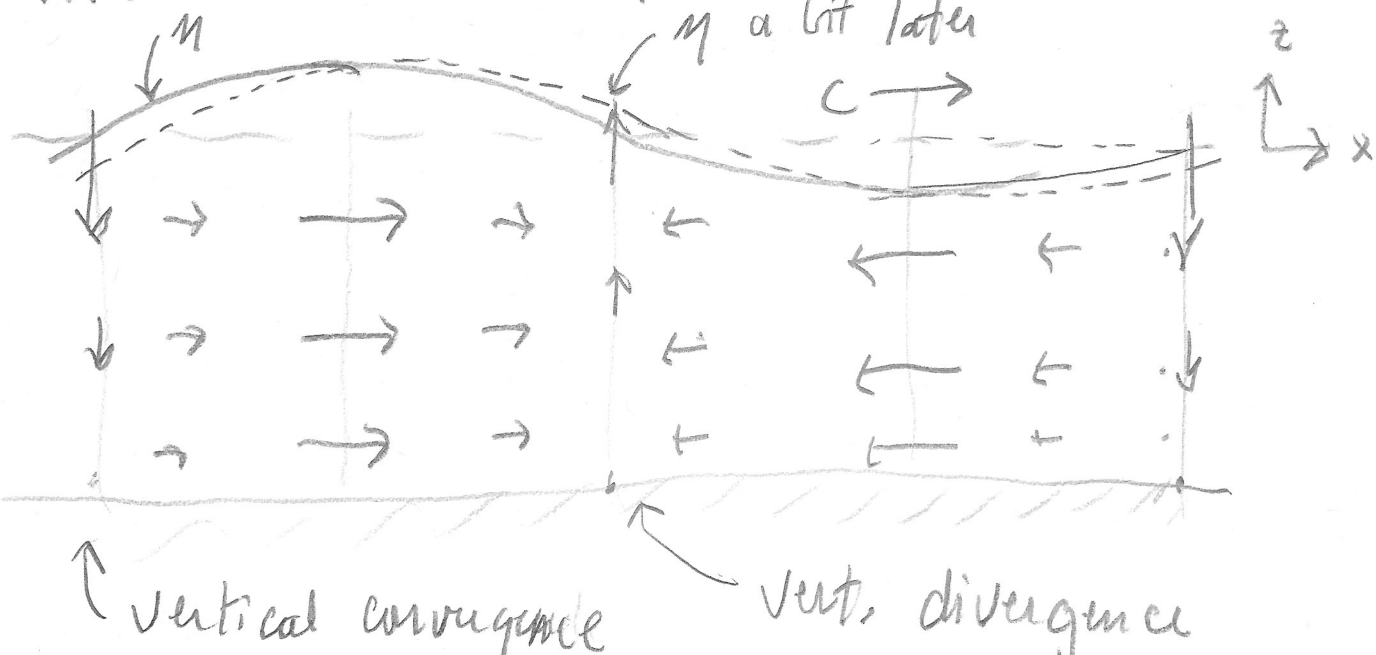
$$u = Re\left\{\frac{\omega}{k}\frac{a}{H} \exp i(kx - \omega t)\right\}$$

Without friction $k = k_0 = \frac{\omega}{c}$ Real

$\Rightarrow$ u in phase with η

and $[u] = c \frac{a}{H}$ $\Rightarrow$ u faster in shallow water $\sim H^{-1/2}$

How to remember pattern:



With Friction, for $R/\omega = 1$

$$k = \frac{\omega}{c}\, 2^{1/4} \exp\left(i\,\pi/8\right)$$

$$\Rightarrow \frac{1}{k} = \frac{c}{\omega}\, 2^{-1/4} \exp\left(-i\,\pi/8\right)$$

(*)

$$\text{so } u = \mathrm{Re}\left\{ c\,\frac{a}{H}\, 2^{-1/4} \exp i\left(kx - \omega t - \pi/8\right)\right\}$$

Just like frictionless solution but

max speed decreased by $2^{-1/4} = 0.85$

and with phase <u>lead</u> (*)

of $\pi/8 \simeq 22\frac{1}{2}^\circ = \frac{12.42h}{16} = .77$ hours

[Physically this is because smaller u has less inertia to overcome so it responds faster to η_x.] !